Seismic anisotropy

Stiffness matrix C_{ij}

$$C_{ij} = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1113} & c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & c_{2223} & c_{2213} & c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & c_{3323} & c_{3313} & c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & c_{2323} & c_{2313} & c_{2312} \\ c_{1311} & c_{1322} & c_{1333} & c_{1323} & c_{1313} & c_{1312} \\ c_{1211} & c_{1222} & c_{1233} & c_{1223} & c_{1213} & c_{1212} \end{pmatrix}$$

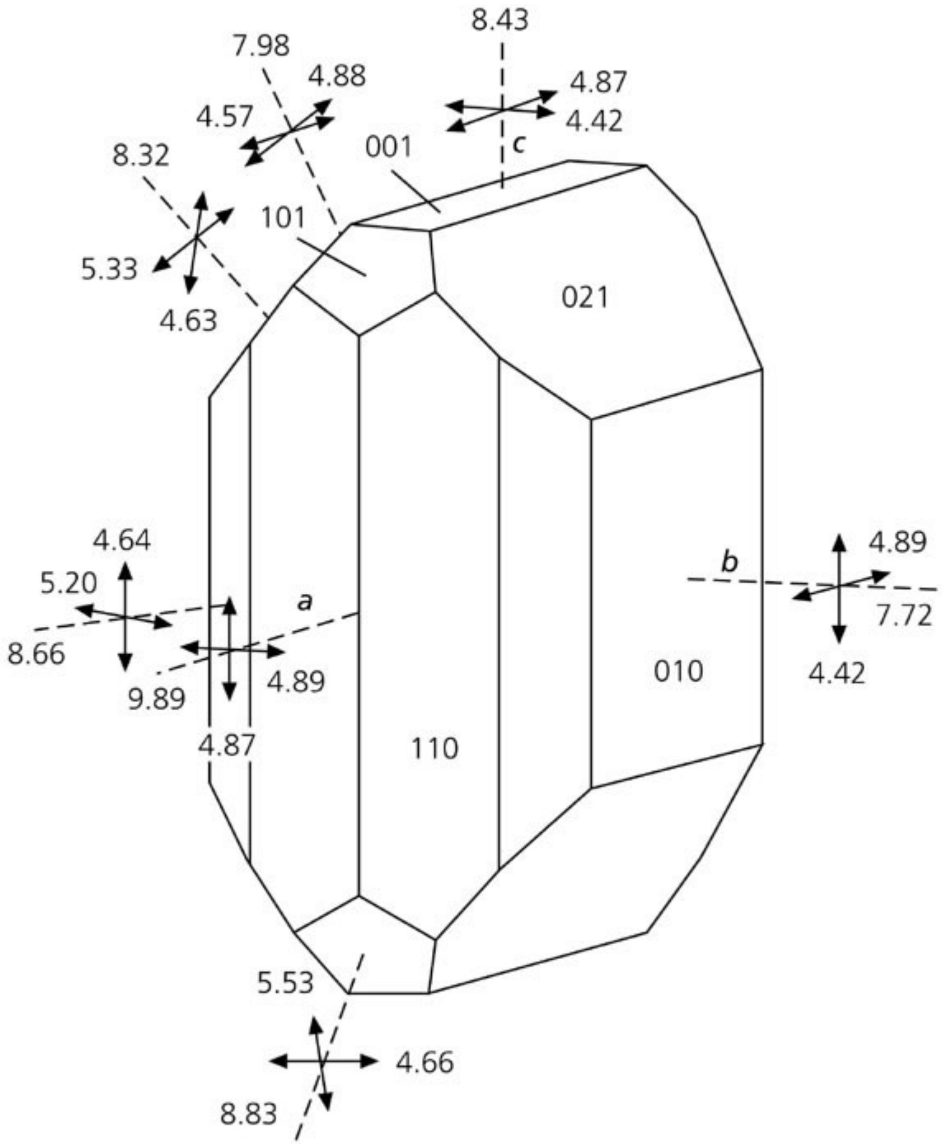
- 4^{th} order tensor
- 21 independent elements
- symmetric

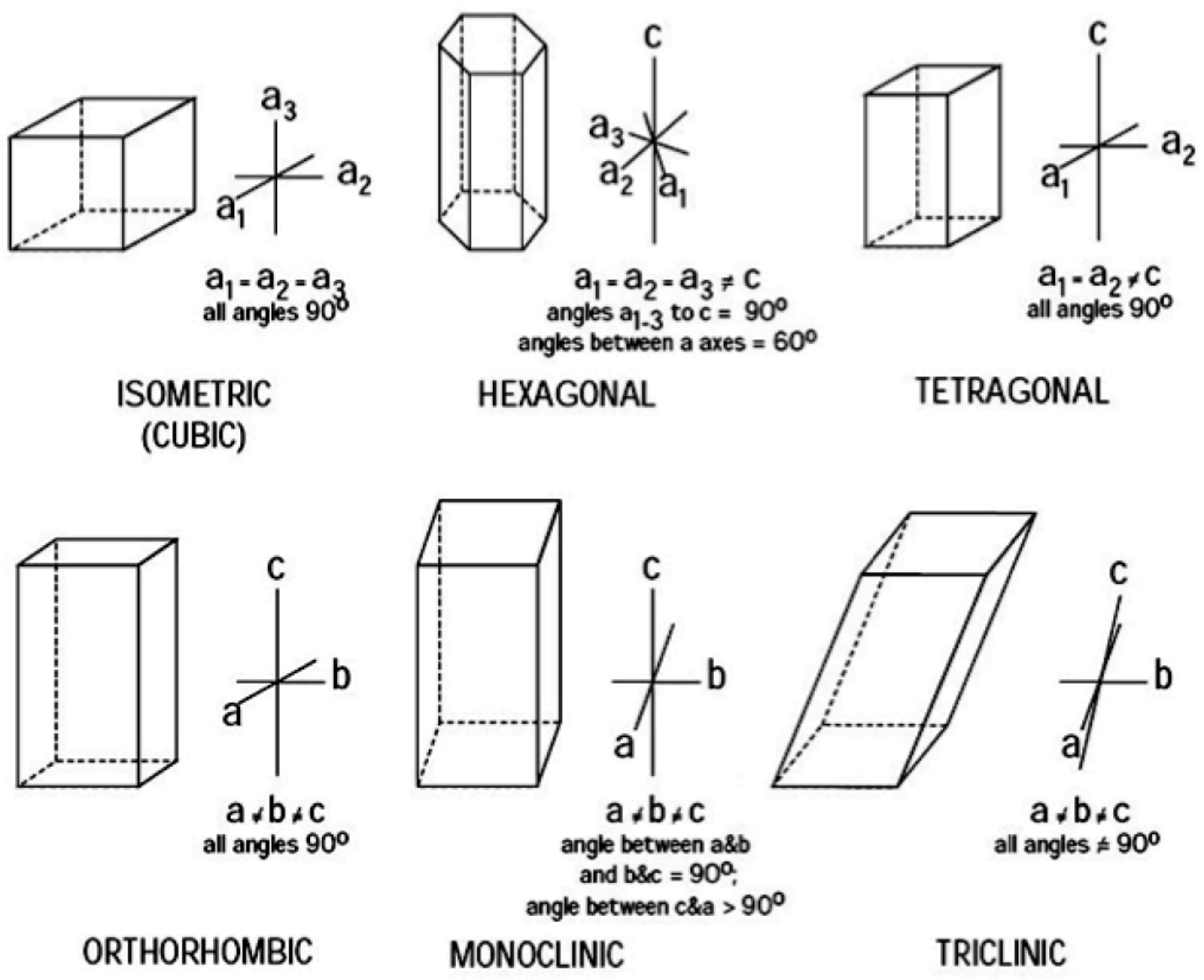
Rules for indices C_{ij} and c_{ijkl}

Pairs of (I,j) and (k,l) $\rightarrow$ index C of c_{ijkl}

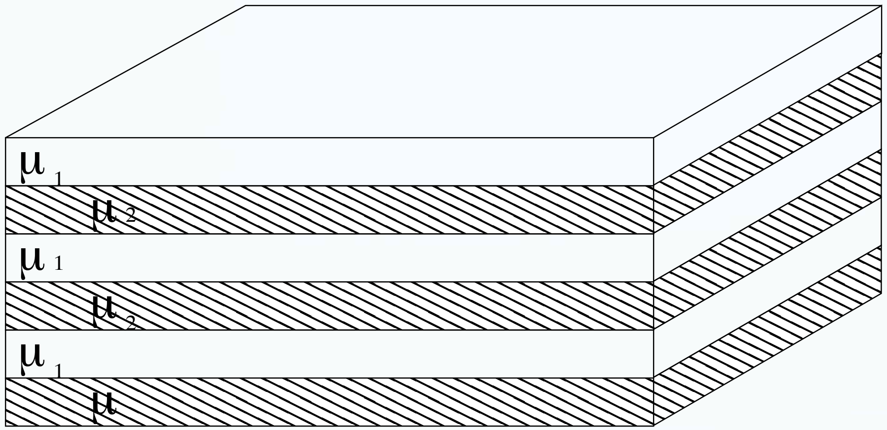
$$\begin{aligned} (1,1) &\rightarrow 1 \\ (2,2) &\rightarrow 2 \\ (3,3) &\rightarrow 3 \\ (2,3) &\rightarrow 4 \\ (1,3) &\rightarrow 5 \\ (1,2) &\rightarrow 6 \end{aligned}$$

Figure 3.6-3: Anisotropy of an olivine crystal.

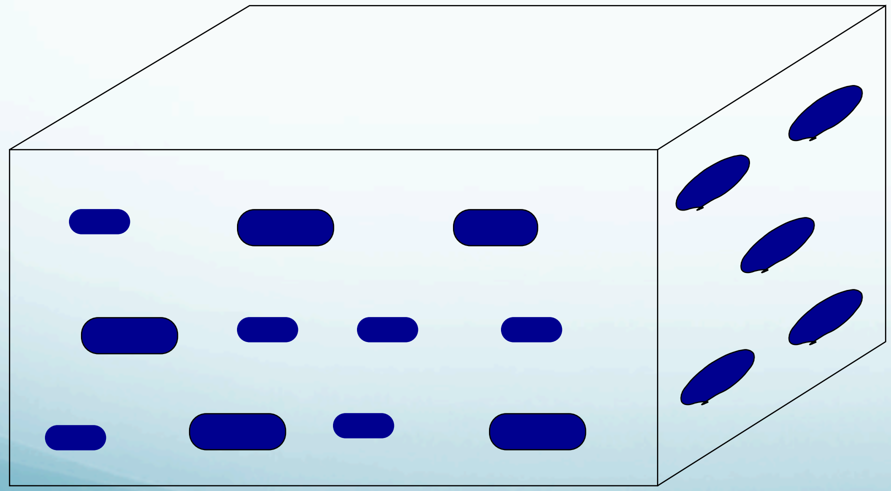



a3
a2
a1
a1 = a2 = a3
all angles 90°
ISOMETRIC
(CUBIC)
c
a3
a2
a1
a1 = a2 = a3 ≠ c
angles a1-3 to c = 90°
angles between a axes = 60°
HEXAGONAL
c
a2
a1
a1 = a2 ≠ c
all angles 90°
TETRAGONAL
c
b
a
a ≠ b ≠ c
all angles 90°
ORTHORHOMBIC
c
b
a
a ≠ b ≠ c
angle between a&b
and b&c = 90°;
angle between c&a > 90°
MONOCLINIC
c
b
a
a ≠ b ≠ c
all angles ≠ 90°
TRICLINIC

Type of symmetry	Number of independent elastic coefficients	Typical mineral
isotropic solid	2	volcanic glass
cubic	3	garnet
hexagonal	5	ice
trigonal I	7	ilmenite
trigonal II	6	quartz
tetragonal	6	stishovite
orthorhombic	9	olivine
monoclinic	13	hornblende
triclinic	21	plagioclase



Stack of isotropic layers



Medium with aligned cracks

Stiffness matrix

Isotropic: 2 independent constraints (volcanic glass)

$$C = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

Hexagonal: 5 independent constraints (ice)

$$C = \begin{pmatrix} A & A-2N & F & 0 & 0 & 0 \\ A-2N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}$$

A, C, L, N, F : Love's elastic constants

Wave propagation in anisotropic media

Wave equation in homogeneous anisotropic medium:

$$c_{ijkl}\partial_j\partial_k u_l = \rho\partial_{tt}u_i$$

Plane wave solution of the form:

$$\vec{u}(\vec{r},t) = \vec{a}f(t - \frac{\vec{n}\cdot\vec{r}}{c})$$

Substitution:

$$c_{ijkl}\partial_j\partial_k[a_l f(t - \frac{\vec{n}\cdot\vec{r}}{c})] = \rho\partial_{tt}[a_i f(t - \frac{\vec{n}\cdot\vec{r}}{c})]$$

$$c_{ijkl}a_l n_j n_k \frac{1}{c^2} = \rho a_i$$

$$\boxed{m_{il}a_l = c^2 a_i}$$

$$\boxed{m_{il} = \frac{1}{\rho}c_{ijkl}n_j n_k}$$

m_{il} = M = Christoffel matrix

1. Isotropic medium

Substituting the isotropic medium stiffness tensor

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

in

$$m_{il} = \frac{1}{\rho} c_{ijkl} n_j n_k$$

Results in

$$m_{il} = \frac{1}{\rho}[\lambda n_i n_l + \mu(n_i n_l + \delta_{il}(n_1 n_1 + n_2 n_2 + n_3 n_3))]$$

$$m_{il} = \frac{1}{\rho}[\lambda n_i n_l + \mu(n_i n_l + \delta_{il})]$$

Giving the Christoffel matrix

$$M = \frac{1}{\rho}\begin{pmatrix} (\lambda+\mu)n_1^2 + \mu & (\lambda+\mu)n_1 n_2 & (\lambda+\mu)n_1 n_3 \\ (\lambda+\mu)n_1 n_2 & (\lambda+\mu)n_2^2 + \mu & (\lambda+\mu)n_2 n_3 \\ (\lambda+\mu)n_1 n_3 & (\lambda+\mu)n_2 n_3 & (\lambda+\mu)n_3^2 + \mu \end{pmatrix}$$

Eigenvalues	$\Lambda_1 = \mu/\rho$	$\Lambda_2 = \mu/\rho$	$\Lambda_3 = (\lambda+2\mu)/\rho$
Eigenvectors <u>a</u>:	$a_1n_1+a_2n_2+a_3n_3=0$		
	($\underline{a}.\underline{n}$)=0		$\underline{a}=\underline{n}$

Stiffness matrix

Isotropic: 2 independent constraints (volcanic glass)

$$C = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

Hexagonal: 5 independent constraints (ice)

$$C = \begin{pmatrix} A & A-2N & F & 0 & 0 & 0 \\ A-2N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}$$

A, C, L, N, F : Love’s elastic constants

2. Hexagonal symmetry

With vertical (x_3) symmetry axis

$$C = \begin{pmatrix} A & A-2N & F & 0 & 0 & 0 \\ A-2N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}$$

Choose $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$m_{il} = \frac{1}{\rho} c_{ijkl} n_j n_k = \frac{1}{\rho} c_{i11l}$$

$$M = \frac{1}{\rho} \begin{pmatrix} c_{1111} & c_{1112} & c_{1113} \\ c_{2111} & c_{2112} & c_{2113} \\ c_{3111} & c_{3112} & c_{3113} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{61} & C_{66} & C_{65} \\ C_{51} & C_{56} & C_{55} \end{pmatrix}$$

$$M = \frac{1}{\rho} \begin{pmatrix} A & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & L \end{pmatrix}$$

Eigenvectors and eigenvalues are …

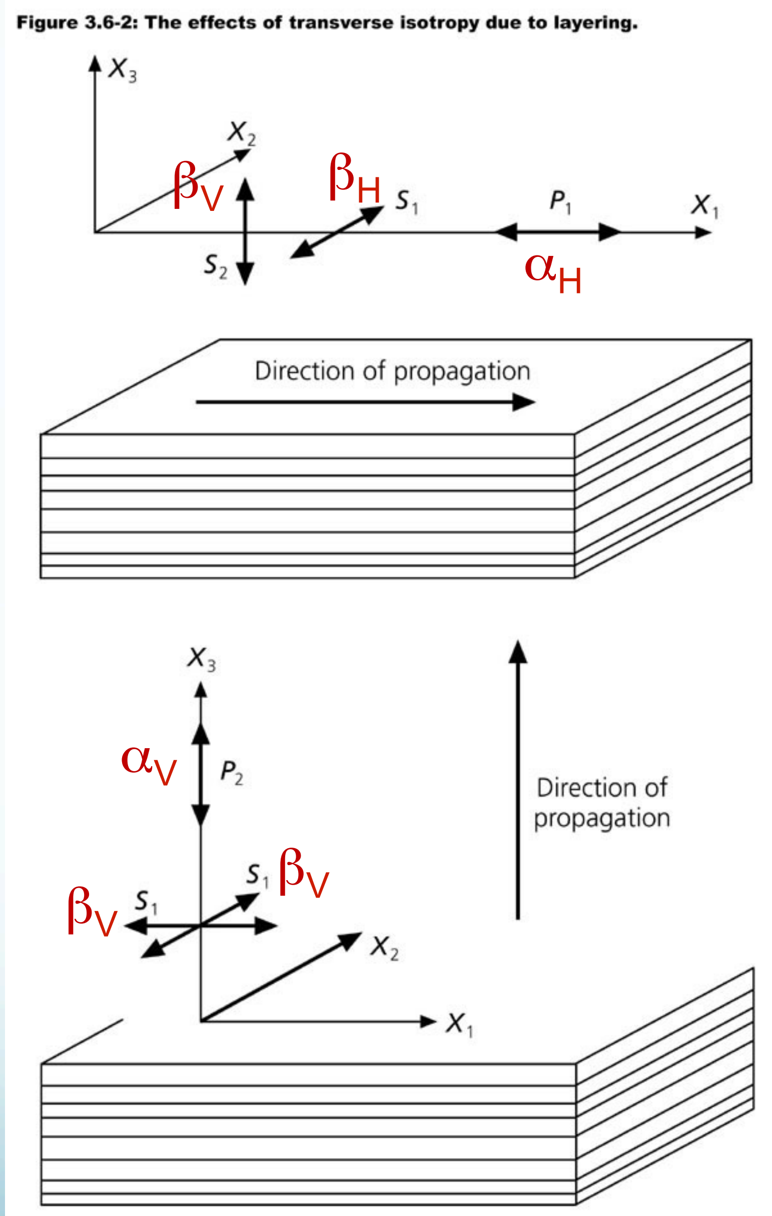
2. Hexagonal symmetry

Choose $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$m_{il} = \frac{1}{\rho} c_{ijkl} n_j n_k = \frac{1}{\rho} c_{i33l} = \frac{1}{\rho} \begin{pmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & C \end{pmatrix}$$

Eigenvectors and eigenvalues are …

$$\alpha_H = \sqrt{\frac{A}{\rho}}$$
$$\beta_H = \sqrt{\frac{N}{\rho}}$$
$$\beta_V = \sqrt{\frac{L}{\rho}}$$



$$\alpha_V = \sqrt{\frac{C}{\rho}}$$
$$\beta_V = \sqrt{\frac{L}{\rho}}$$

Layering:

A > C $\alpha_H > \alpha_V$

N > L $\beta_H > \beta_V$

Weak anisotropy

We get quasi P- and S-waves:

$$\rho V_P^2 = A + B_c \cos 2\theta + B_s \sin 2\theta + C_c \cos 4\theta + C_s \sin 4\theta$$

$$\rho V_{SH}^2 = D + E_c \cos 4\theta + E_s \sin 4\theta$$

$$\rho V_{SV}^2 = F + G_c \cos 2\theta + G_s \sin 2\theta$$

Surface waves:

$$c(\omega, \theta) = c_0(\omega) + c_1(\omega) \cos 2\theta + c_2(\omega) \sin 2\theta + c_3(\omega) \cos 4\theta + c_4(\omega) \sin 4\theta$$

Surface waves – group and phase velocity

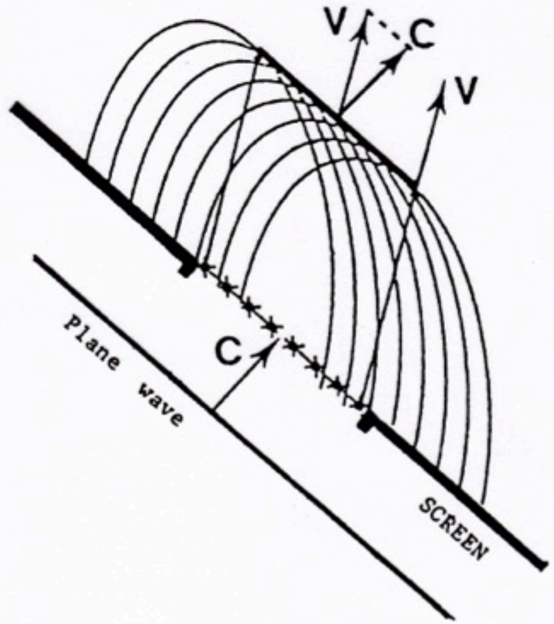


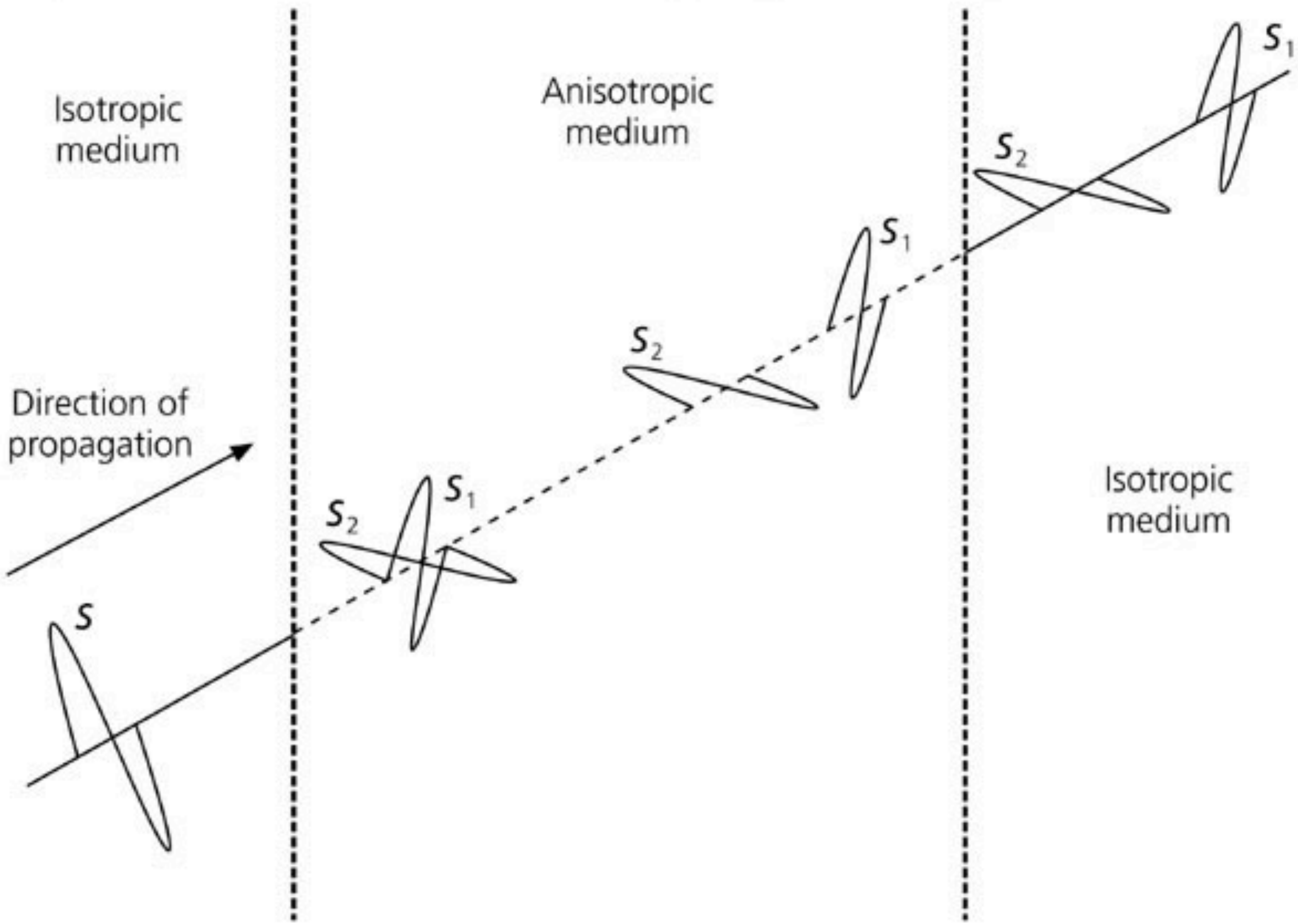
Figure 2-5 Illustration of the concept of group velocity **v** and phase velocity **c** in anisotropic media (modified from Garmany, 1989). The phase velocity vector **c** is perpendicular to the phase surface while the group velocity vector **v** is parallel to the direction of the beam of seismic energy.

Seismic observations of anisotropy

- Shear wave splitting (S-wave birefringence)
- Love/Rayleigh incompatibility
- SH-derived models of the upper mantel have higher velocities than P-SV derived models (body waves)
- Pn velocities are azimuth dependent
- Travel times PKIKP waves: equatorial paths slow, polar paths fast

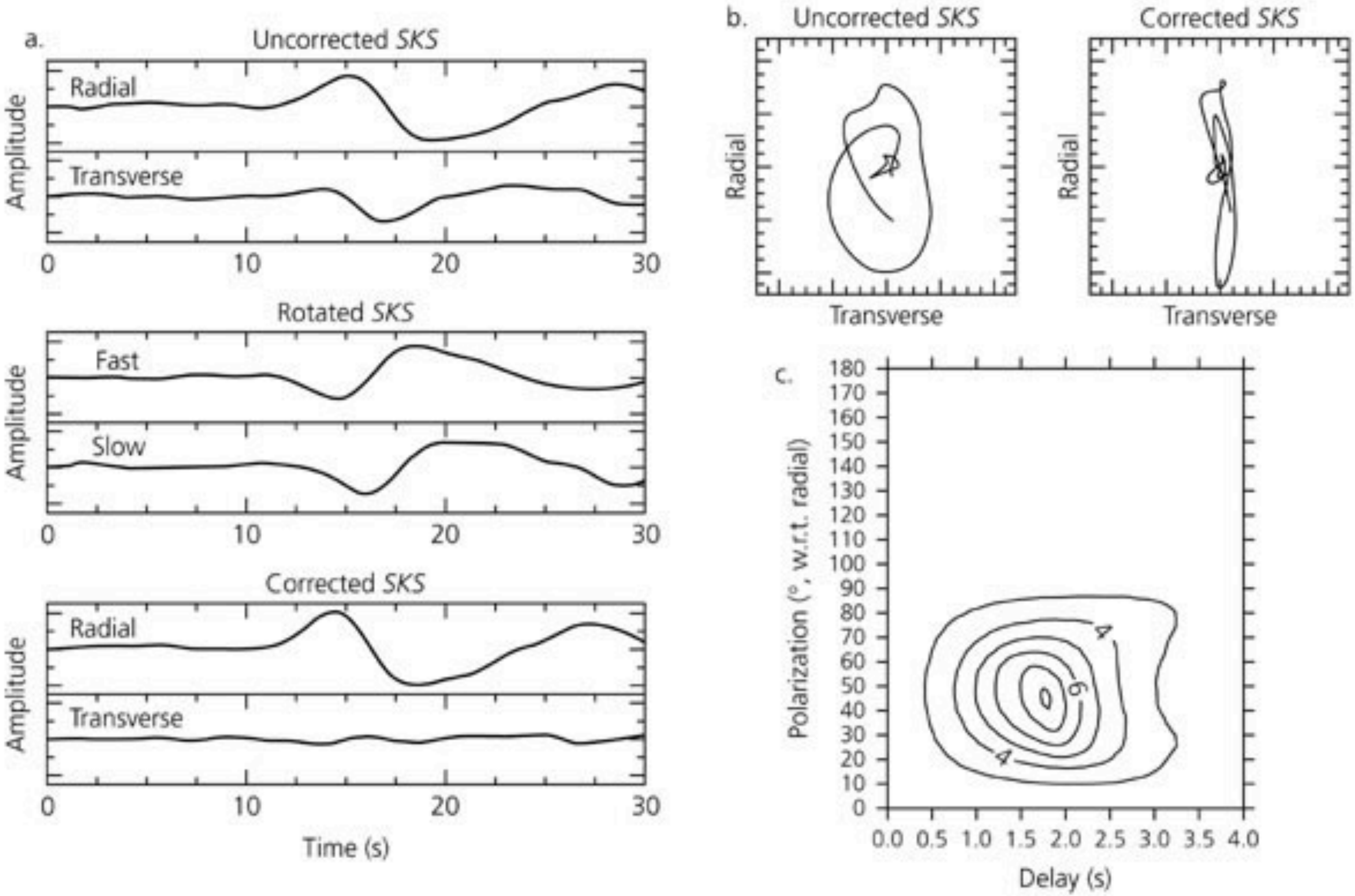
Shear wave splitting

Figure 3.6-1: Cartoon of a shear wave split by an anisotropic medium.

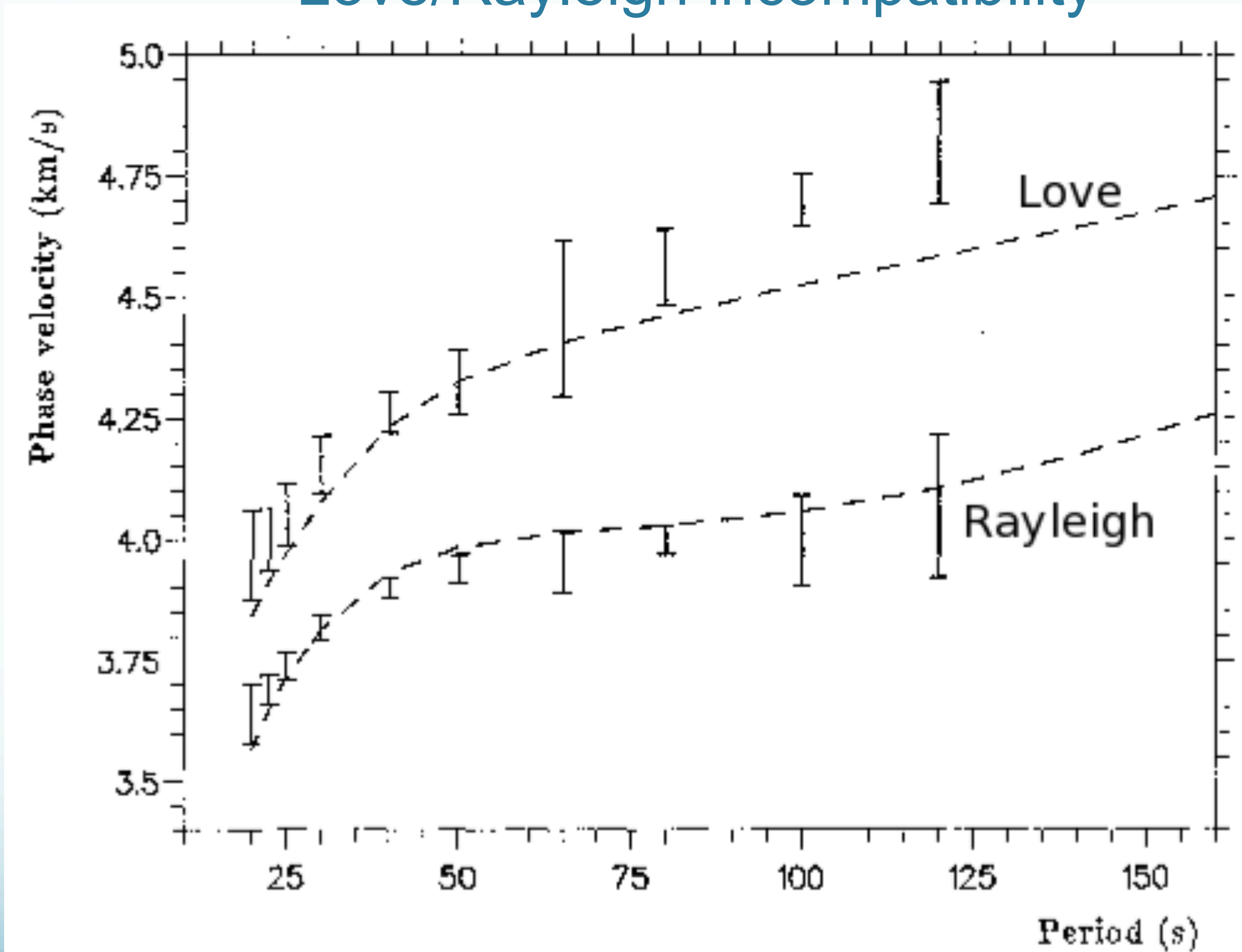


Shear wave splitting

Figure 3.6-7: Example of shear wave splitting of *SKS* waves.

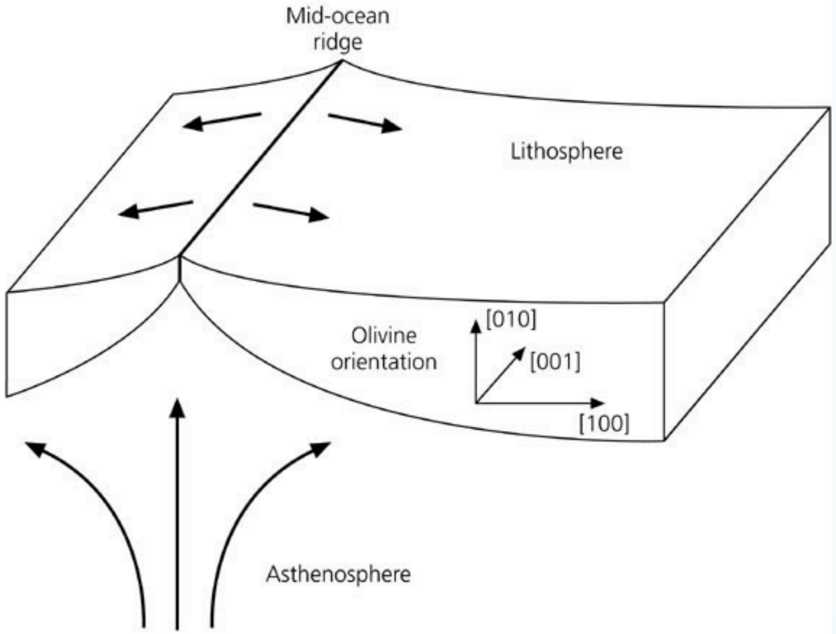


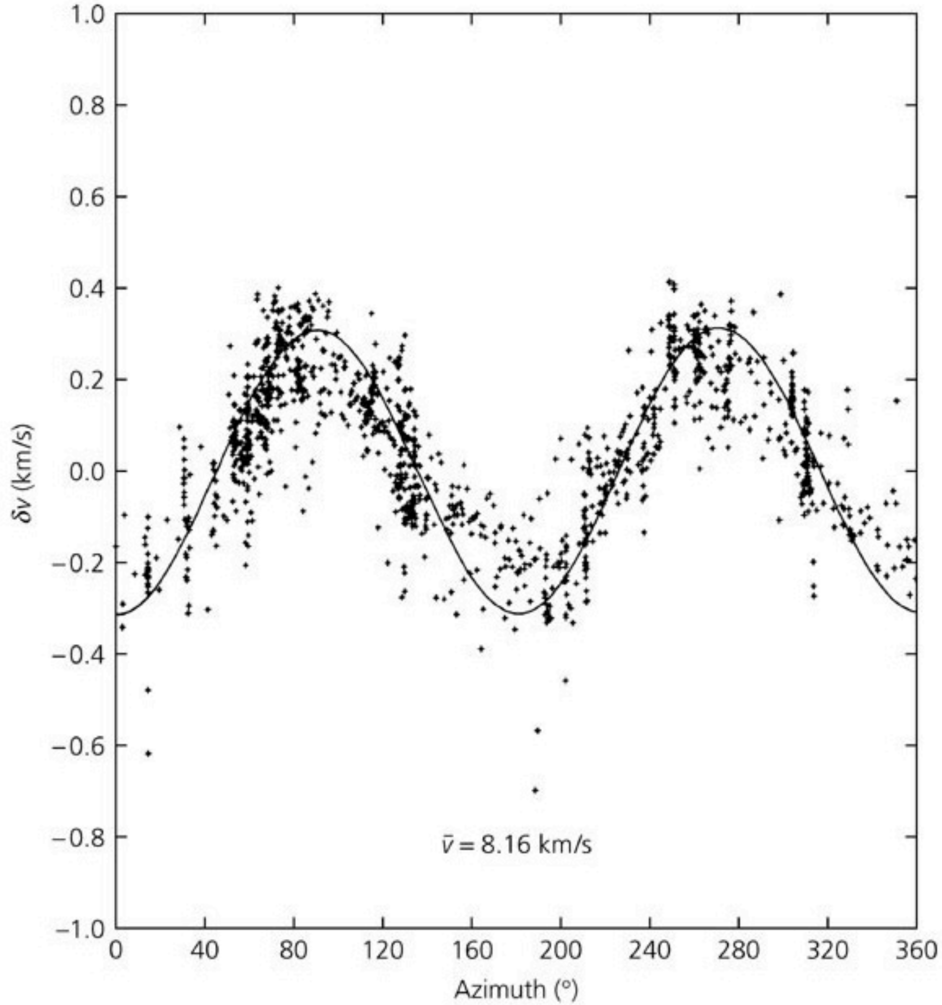
Love/Rayleigh incompatibility

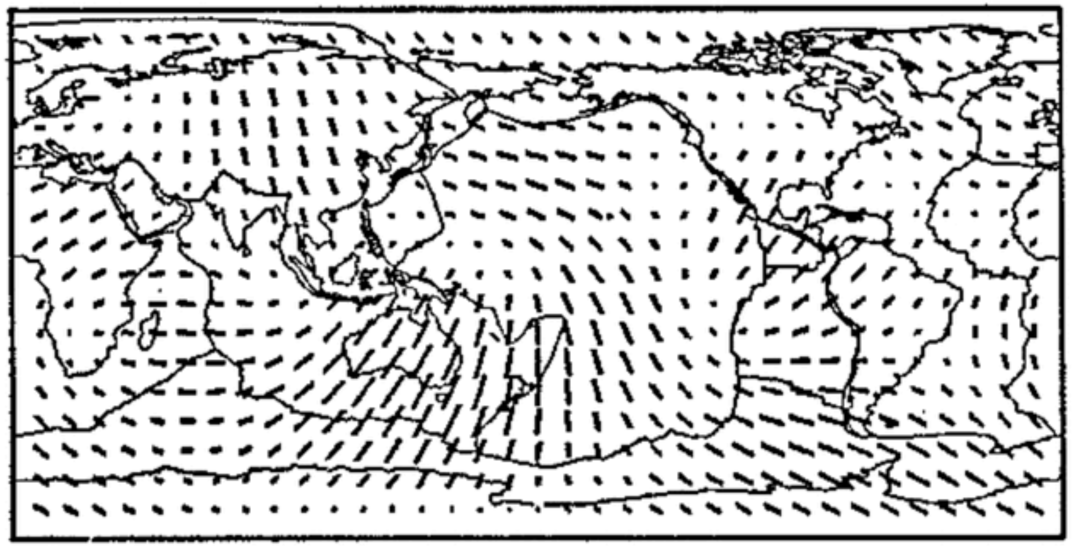


Pn waves azimuthal anisotropy

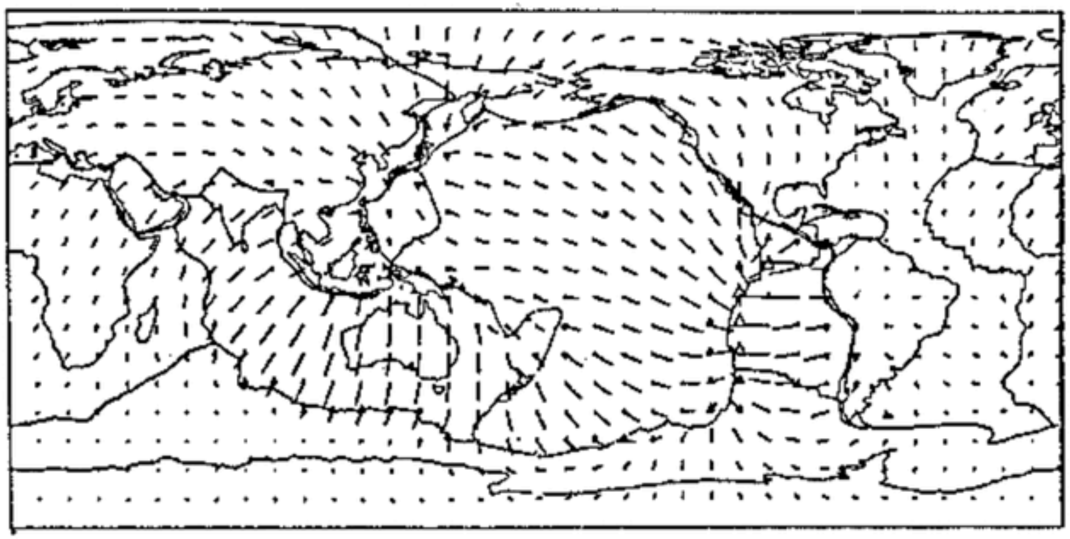
Figure 3.6-4: Example of anisotropy in the oceanic lithosphere.





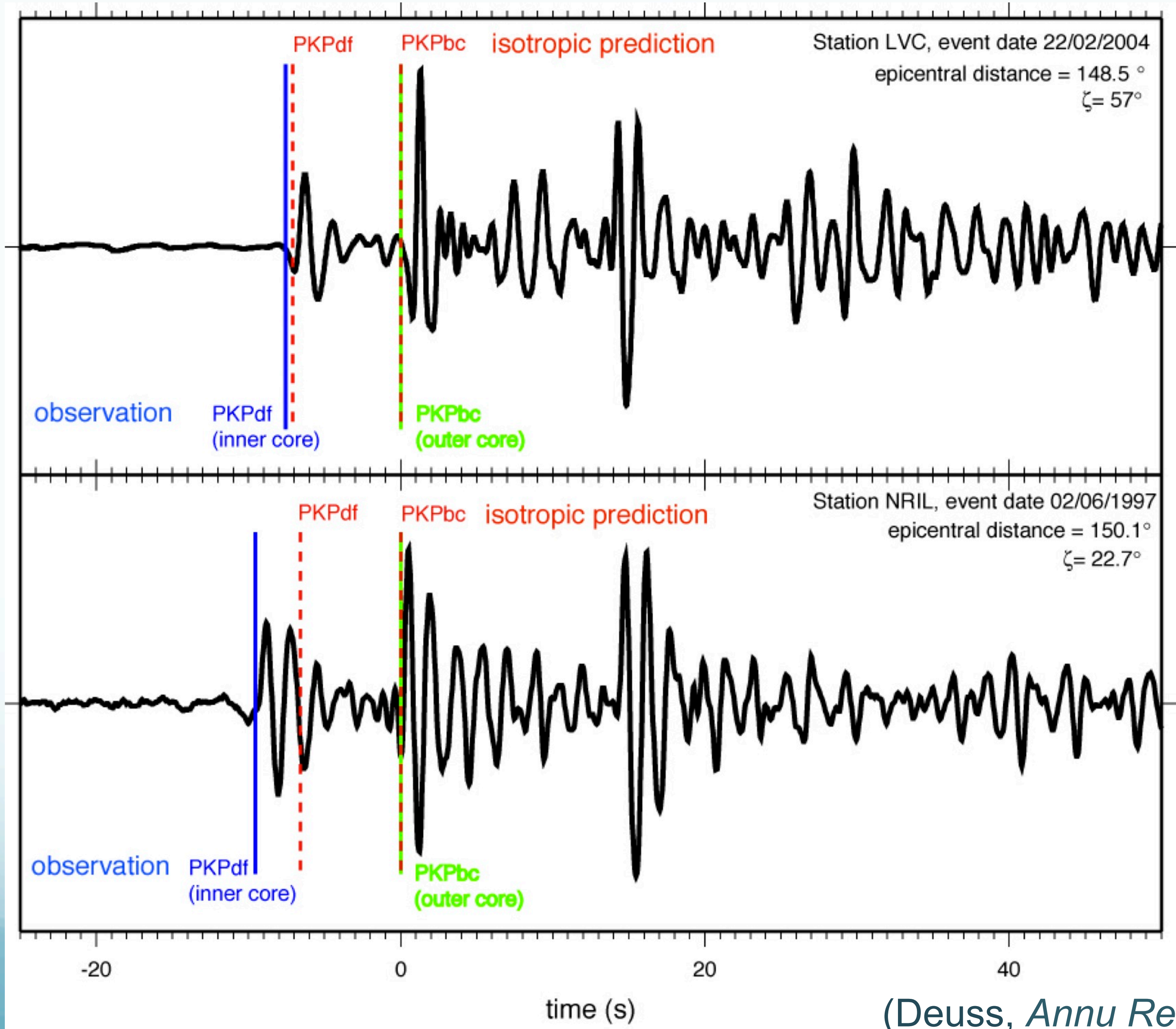


(a) Rayleigh wave azimuthal anisotropy



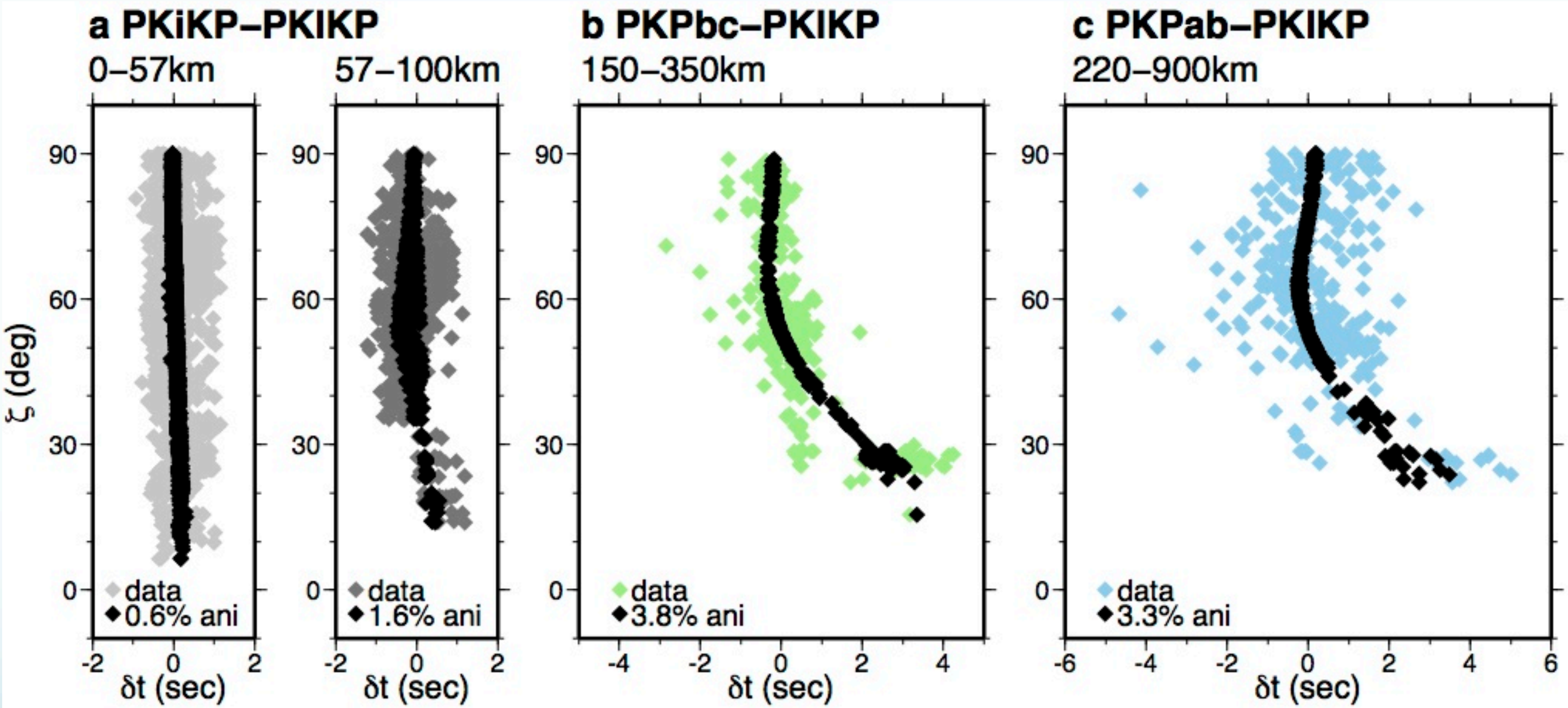
(b) Model of upper mantle flow directions

PKIKP wave travel times



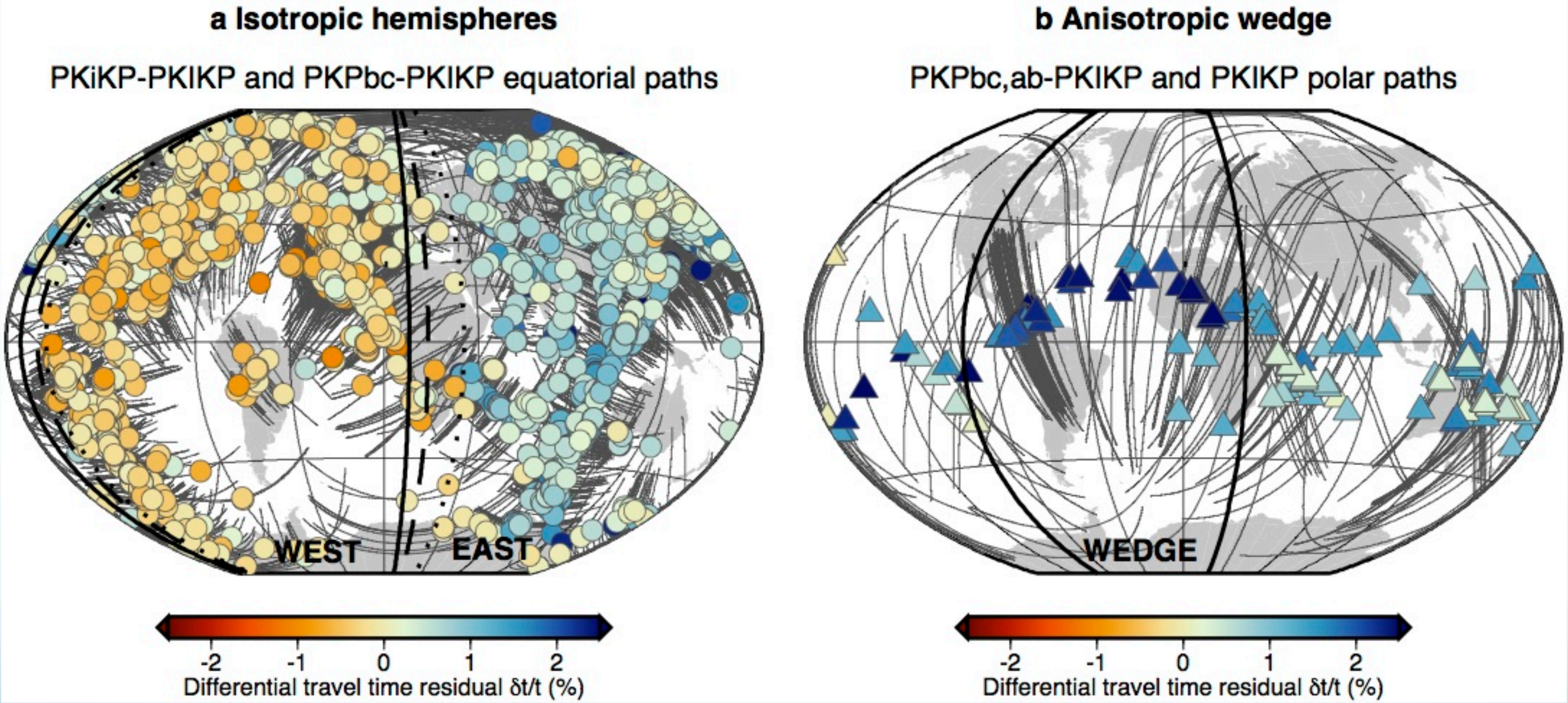
(Deuss, *Annu Rev.*, 2014)

PKIKP wave travel times



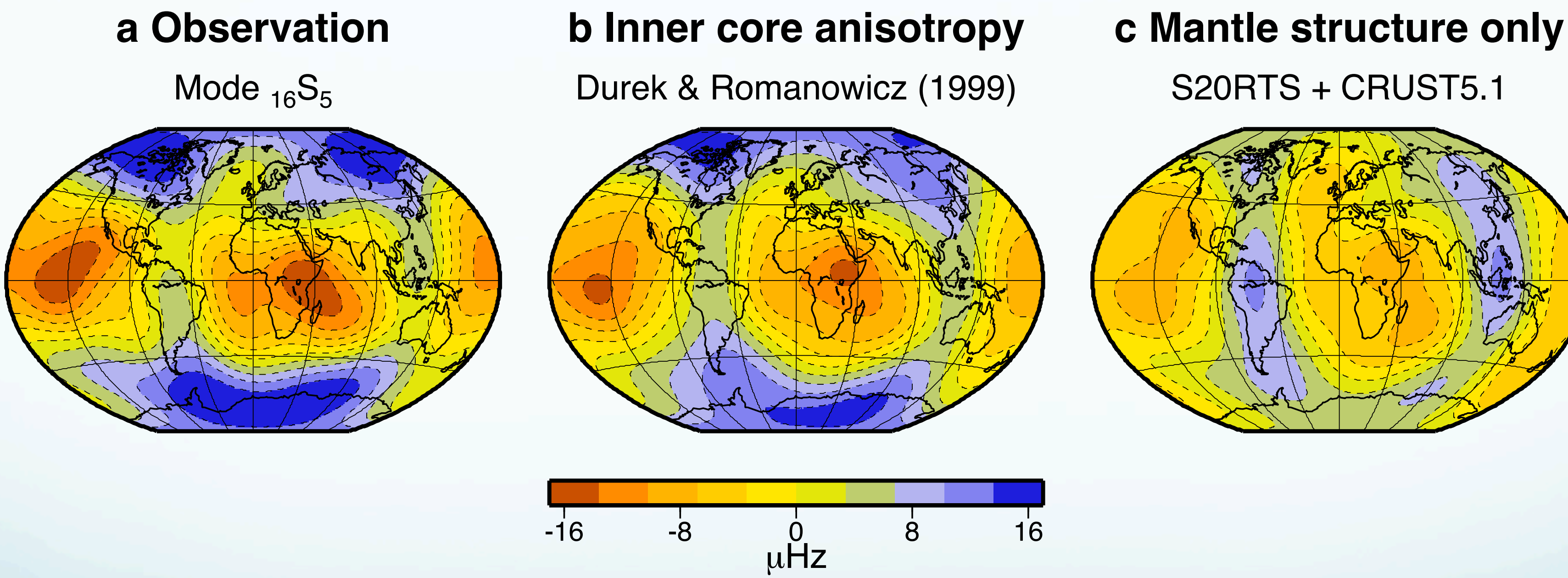
(Deuss, *Annu Rev.*, 2014)

PKIKP wave travel times

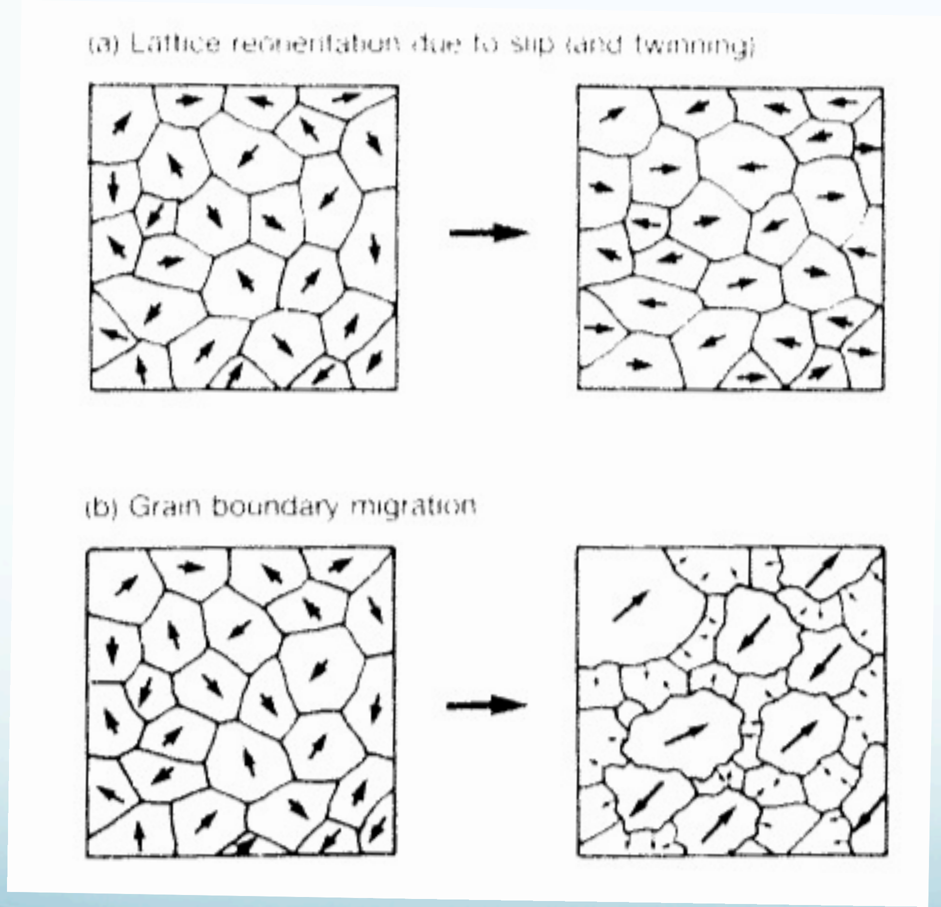


(Deuss, *Annu Rev.*, 2014)

Normal mode splitting functions



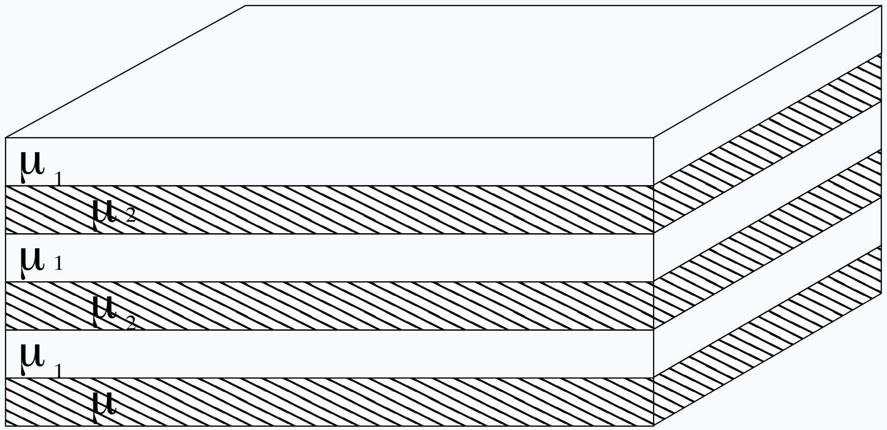
Lattice preferred orientation (LPO)



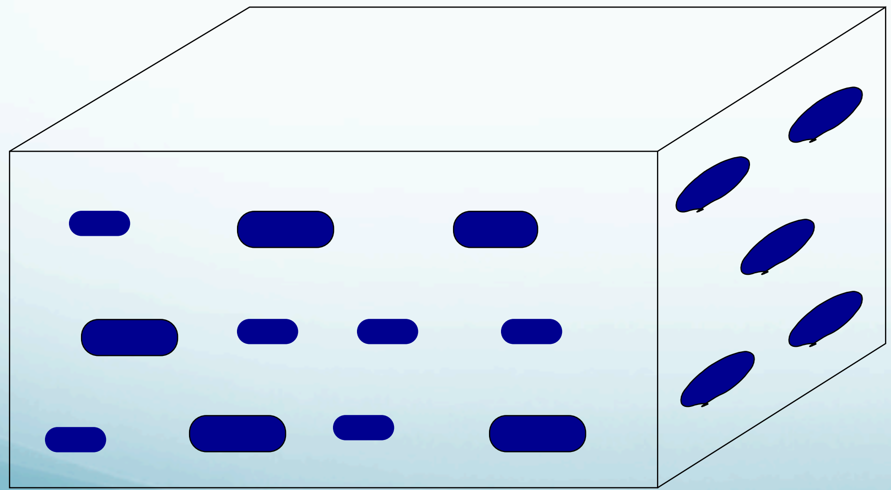
(a) Dislocation glide

(b) Grain boundary migration

Anisotropic shape distribution of isotropic materials



(a) Stack of isotropic layers



(b) Medium with aligned cracks